Game Theory 博弈論- John Nash

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A study of making the best beneficial strategy according to the opponents' moves in a non-cooperative game 博弈遊戲.

It is very useful in doing business planning and can explain many social behaviours.

For examples,

- When economy going downturn, big merchandizing companies are considering cutting prices and putting products on sale. Could that be avoidable?
- How to raise the chances of making crime suspects to confess?

A case - Prisoner's Dilemma 囚徒困境

Two prisoners A and B committed a crime. Without no evidence, during the investigation, they are both questioned individually and separately. They are not allowed to discuss with each other. During the questioning, they can either confess and become a witness or refuse to confess. If both confess, they will each sentence to 8 years of imprisonment. If either one confesses and become a witness. As a reward, the witness will not be punished. The other one will receive 10 years of imprisonment. If both do not confess, since without no enough evidence, both will only receive 1 year of imprisonment.

The sentences are summarized and listed on below:

	B confesses	B does not confess
A confesses	-8, -8	0, -10 (A has no imprisonment, B has 10 years of imprisonment)
A does not confess	-10, 0 (A has no 10 years of imprisonment, A has no imprisonment)	-1, -1

From the above, it is cleared that from either A's or B's personal point of view, they will try to confess in order to make no imprisonment. But they will both end up with 8 years of imprisonment.

If globally speaking, they both do not confess, they will have only 1 year of imprisonment.

Hence, according to the theory, it is obvious that personal benefit may not achieve global benefit.

This explains a lot of social phenomena because everyone is living for the best benefits of its own, thus resulting not the best benefit for the society. It only be solved, through laws to regulate individual behaviors.

For example, all people want to enjoy more freedom. But that will result in someone's freedom is being eroded by others. Hence, we must though laws to limit the individual freedom under certain conditions.

Say, some people want to sing at day and night. That will infringe the freedom of others enjoying silence. Only with laws to control people lower their sounds at night while allowing them to sing in daylight, then both parties could enjoy live.

Other examples may be on sharing the social resources -- people want to get as much resources as possible but that will jeopardize others' shares. Hence it must through careful rules to control the behaviors or rely on social-moral ethics or self-religious believes.

An exercise – Price Reduction War

Two competing companies A and B are considering to have an on-sale campaign to promote sale. If both companies are not on-sale, they will both earn 2 million. But if one company is on-sale and the other not, then the on-sale company will earn 3 million and the non-on-sale company will only earn 0.5 million. If both are on sale, they will both earn 1 million.

Fill up the profit table below, some of them are filled up for you.

	B on-sale	B not on-sale
A on-sale		3, 0.5
A not on-sale		2, 2

Answer:

	B on-sale	B not on-sale
A on-sale	1, 1	3, 0.5
A not on-sale	0.5, 3	2, 2

Profit of A company: 3 million (A on-sale, B not on-sale) > 2 million (both not on-sale) > 1 million (both on-sale) > 0.5 million (A not on-sale, B on-sale)
Profit of B company: 3 million (A not on-sale, B on-sale) > 2 million (both not on-sale) > 1 million (both on-sale) > 0.5 million (A on-sale, B not on-sale)

According to the profit table above, draw your conclusions on their final movements.

Answer:
Since both companies are clear about the situation, they must understand if they do nothing (not on-sale), they would only be benefited when the opponent is also not on-sale
But if the opponent puts on-sale, they will lose in the game. In order to be taking control of the market, they must act and puts products on-sale. As a result, in return, the opponent must follow to put on-sale. Thus, it is unavoidable to have a price-cutting war.

A case – Smart Pigs Choices 智豬博弈

Two very clever smart pigs, one big and one small, are standing by a eating device for their meal. But they must turn on a switch which is located far away from the device in order to have the meal. While a pig goes away for turning on the switch, the meal will soon arrive at the device and the staying behind pig will immediately take up his share.

If the total number of shares is 10.
If big pig has meal first, big pig will take up 9 shares and small pig will take up 1 share.
If small pig has meal first, big pig will take up 6 shares and small pig will take up 4 shares.
If both pigs have meal together, big pig will take up 7 shares and small pig will take up 3 shares.

Also, if either pig goes to turn on the switch, both pigs waste 2 shares of energy of the travel trip.

In summary of the above:

	Small pig goes to turn on the switch	Small pig waits by the device
Big pig goes to turn on the switch	7-2, 3-2 (both eat together) = 5, 1	6-2, 4-0 (big pig eats first) = 4, 4
Big pig waits by the device	9, 1-2 (big pig eats first) = 9, -1	(no one turns on switch) = 0, 0

(Note: 5, 1 means big pig gets 5 shares, small pig gets 1 share respectively)

According to the table, it is clear that big pig can take up 9 shares if he waits (max), or 0 share (min) if he goes, depending on the moves of the small pig. So it is not certain for him to go or not go.

But by looking at the right column in the summary table, for the small pig, it is clear that he should wait because the result will be 4 or 0 which are both larger than 1 or -1 (the left column, if he goes). In other words , it is quite sure for the small pig that no matter on what decision of the big pig, he must stay (wait / not go) by the device.

In return, since finding out if the small pig waits, then the big pig must choose to go and turn on the switch otherwise he has nothing to eat.

I.e. the final result should be both pigs have 4 shares (i.e. 4, 4) when the big pig goes to turn on the switch and the small pig waits.

The above applies to many situations with different benefits for different participants.

Zero-sum Game 零和博弈 / 零和遊戲

It is a situation which involves two sides, where the result is an advantage for one side and an equivalent loss for the other. That means when one side wins, the other side loses and which is not a win-win situation. There should be a balance point (a most probable decision between participants) inside a Zero-sum Game. We can find that out by calculations.

A Three-person Game

A, B, C are playing a game in which they could either put up one-finger or two-fingers. The only person who put up one-finger will get 1 score point. The only person who put up two-fingers will get 2 score points. The other cases will get no score point.

It is quite clear that in order to get more score points, everyone should put up two-fingers – even though if others also put up two-fingers, they will end up with no score point. But what exactly the probability will they do that?

C One-finger (r)		B	
		One-finger (q)	Two-fingers (1-q)
A	One-finger (p)	0, 0, 0	0, 2, 0
	Two-fingers (1-p)	2, 0, 0	0, 0, 1

C Two-fingers (1-r)		B	
		One-finger (q)	Two-fingers (1-q)
A	One-finger (p)	0, 0, 2	1, 0, 0
	Two-fingers (1-p)	0, 1, 0	0, 0, 0

Given that
Probability of A who puts up one-finger = p, where $0 < p < 1$
Probaility of A who puts up two-fingers = 1-p
....

A's expectation score point = $2(1-p)qr + 1p(1-q)(1-r) = [(1-q)(1-r)-2qr]p+2qr$
B's expectation score point = $2p(1-q)r + 1(1-p)q(1-r)$
C's expectation score point = $1(1-p)(1-q) + 2pq(1-r)$

It is clear that the above three score points must be the same. That is $p = q = r$. And for the maximum expectation score point of A is when $[(1-q)(1-r)-2qr] = 0$, hence by calculation (when $p = q = r$) => sqr(2)-1 = 0.414, and the max score point = 0.3428.
I.e. If the game repeats many times, each participant will on average get 0.3428 point in each game.

Therefore, A, B or C will have 41.4% to put up one-finger, or 58.6% to put up two-fingers. This is in consistence with the above prediction.

A voting game

Generally, the majority wins in a vote.

Three persons A, B, C are having a vote of either pro or against a statement. If their vote is the same as the result, they get 1 point, otherwise no point. Say, if A votes Pro, and the final majority of votes is also Pro. Then he wins 1 point in this vote. If A votes Pro, but no other one votes Pro, the final majority is Against, then he has no point in this vote.

The points are listed on below, fill up the empty ones:

C Pro (r)		B	
		Pro (q)	Against (1-q)
A	Pro (p)	1, 1, 1	
	Against (1-p)		1, 1, 0

C Against (1-r)		B	
		Pro (q)	Against (1-q)
A	Pro (p)		
	Against (1-p)		1, 1, 1

Answer:

C Pro (r)		B	
		Pro (q)	Against (1-q)
A	Pro (p)	1, 1, 1	1, 0, 1
	Against (1-p)	0, 1, 1	1, 1, 0

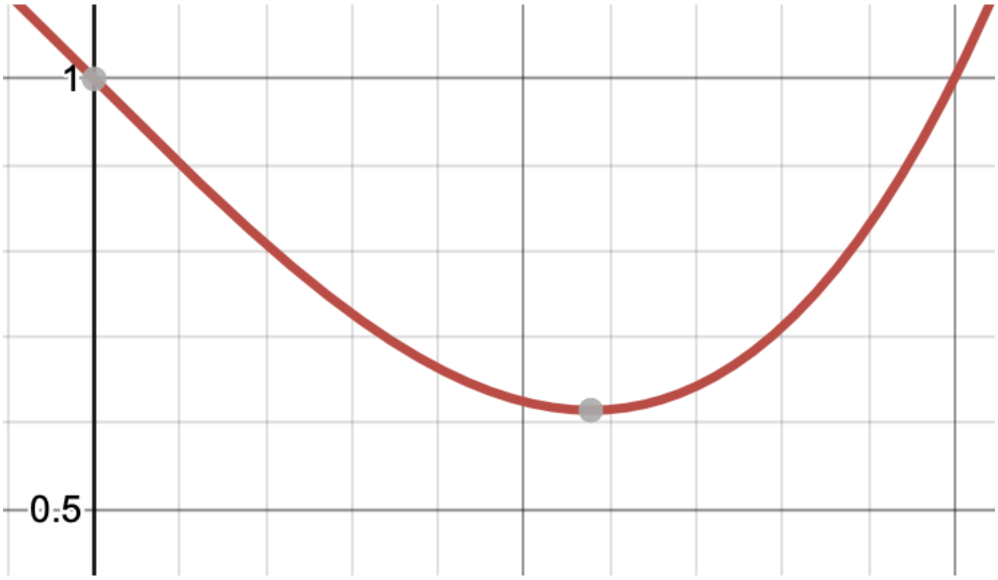
C Against (1-r)		B	
		Pro (q)	Against (1-q)
A	Pro (p)	1, 1, 0	0, 1, 1
	Against (1-p)	1, 0, 1	1, 1, 1

Given that
Probability of A votes Pro = p, where $0 < p < 1$
Probability of A votes Against = 1-p
....

A's expectation score point = $pqr + p(1-q)r + (1-p)(1-q)r + pq(1-r) + (1-p)q(1-r) + (1-p)(1-q)(1-r) = 1 - p - qr + rp + pqr$, if $p=q=r \Rightarrow 1 - p + p^3$

1 (when p = 0 or 1) > The score point > 0.6151 (when p = 0.577)

If we plot p (horizontal axis) ($0 \le p \le 1$), against the score point (vertical axis):



What does it mean by p = 0 or p=1? Also, how do we plan the votes in order to win this game?

Answers:

The maximum score point of A will occur when p = 0 or (1-p) = 1 (A constantly votes only on Against) or p = 1 (A constantly votes only on Pro). That implies if A, B or C wants to get the best score point when they are constantly voting on the same choice. Only when they could not score a point in a game, then they should consider changing the vote to another side.